Exercise 5: Spectral Analysis with the Discrete Fourier Transform (DFT / FFT)

For a successful conduction of the Laboratory Exercise an accurate **preparation** at home is **mandatory**. Otherwise the **participation** in the lab will be **refused** and a repetition of the exercise is required!

1. Introduction and Preparation (homework)

The spectral analysis, also known as spectrometry, is used to determine the amplitude and phase of a signal in terms of frequency. Known from the lecture the spectral analysis of the given signal can be computed by the Discrete Fourier Transform respectively by its more efficient computing algorithm, the FFT (Fast Fourier Transform).

For the **preparation** recapitulate the theory about the DTFT and DFT from the corresponding lecture notes and if required, with the help of literature[1].

In the practical part we will generate elementary signals with Matlab, whose spectra will be observed and evaluated using the Matlab command *fft( )*.

Common terms such as Continuous / Discrete Time (C-T, D-T), Time Limited Signals, Sampling Theorem, Sampling Frequency, Number of Samples, Zero Padding, „Leakage Error“ or „Smearing Error“, Aliasing, Windowing have to be known for the laboratory exercise.

Make hand-written comments for the following questions at home:

1.1	Find the DFT specific relations between • Sampling rate, • Number of samples N (per record), • Time spacing of consecutive samples (sampling period), • Time duration of a whole record, • Zero padding (to Nzp samples)
1.2	Explain the connection between DTFT, DFT and FFT.
1.3	How can the normalized spacing of consecutive spectral lines $\Delta\Omega$ be calculated in DFT respectively FFT? Calculate the resulting spacing of consecutive spectral lines in Hz.
1.4	Determine the DTFT of the following DT signals (see lecture notes). Characterize the spectra of the cosine signals (infinite and finite duration) with help of Eulers's formula. $x_1[n] = e^{j\Omega_0 n}$ for $-\infty < n < \infty$ **and** $0 \le n \le (N-1)$ $x_2[n] = \cos(\Omega_0 n)$ for $-\infty < n < \infty$ **and** $0 \le n \le (N-1)$

[1] Kamen, Heck: Signals and Systems. Pearson Prentice Hall 2007, Third Edition, p. 166-194

2. Conduction of the exercise (must be done in the lab to grade contribution)

For this laboratory exercise only a desktop PC with Matlab is required. The provided m-file *"Training_FFT.m"* generates signals, calculates their spectra and displays the time signal and spectra. The time signal to be analyzed consists of a constant (or DC) component, two sinusoids and noise:

$$x(t) = a_0 + a_1 \cdot \cos(2\pi f_1 \cdot t + \phi_1) + a_2 \cdot \cos(2\pi f_2 \cdot t + \phi_2) + a_n \cdot n(t)$$

The DC component is defined by the coefficient a_0 while the amplitudes, frequencies and phases of the 2 sinusoids are determined by the parameters a_i, f_i, ϕ_i with $i = 1, 2$. The noise is controlled by the coefficient a_n.

For the efficient FFT algorithm it is required that the number of samples N are always a power of 2, (i. e. N = 64, 128, 256, 512, …). When we use a rectangular window and divide the amplitudes of the FFT by the number of samples N (see code in the m-file), the resulting amplitude spectrum shows roughly the half of the real magnitudes (at least with a large number of N). The amplitude of DC component is approximately equal to the real one – that is an important insight for the interpretation of spectra (see conduction of the exercise).

At first copy the provided m-files *"Training_FFT.m"* and *"Test_Signal.mat"* (see folder "*Templates*") to your local Matlab workspace

C:\Users\<YOUR_USERNAME>\Projects\Matlab_2015a

Save your data from the local workspace to an external drive (e. g. Pendrive) or your Home-Drive after each lab training!

Exercise 2.1

Use the following parameters:

a_0	a_n
0.0	0.0

a_1	**f_1 [Hz]**	Φ_1 [rad]	a_2	f_2 [Hz]	Φ_2 [rad]
1.0		0.0	0.0	0.0	0.0

N	Fs [kHz]
64	10

Questions:

2.1.1	Choose the frequency f_1 so that a record of N samples contains exactly 4 periods of this signal component.
2.1.2	Print and discuss the result.
2.1.3	Which is the spacing Δf of the spectral lines?
2.1.4	For which frequencies are spectral lines visible?

Exercise 2.2

Use the following parameters:

a_0	a_n
0.0	0.0

a_1	**f_1 [Hz]**	Φ_1 [rad]	a_2	f_2 [Hz]	Φ_2 [rad]
1.0		0.0	0.0	0.0	0.0

N	Fs [kHz]
64	10

Questions:

2.2.1	Now determine the frequency f_1 so that a record of N samples contains exactly 4.5 periods of the signal component.
2.2.2	Print and discuss the result.
2.2.3	Which is the spacing Δf of the spectral lines?
2.2.4	Which frequencies are represented by the both highest spectral lines located nearby? Mark the corresponding frequency f_1 on the frequency axis.

Exercise 2.3

Use the following parameters:

a_0	a_n
-1.0	0.0

a_1	f_1 [kHz]	Φ_1 [rad]	a_2	f_2 [kHz]	Φ_2 [rad]
2.0	1.2	0.0	1.0	1.3	0.0

N	Fs [kHz]
64	10

Questions:

2.3.1	Print and discuss the result.
2.3.2	Which is the spacing Δf of the spectral lines?
2.3.3	Which frequency is represented by the highest spectral line?
2.3.4	Mark the corresponding frequencies f_1 and f_2 on the frequency axis.

Exercise 2.4

Use the following parameters:

a_0	a_n
-1.0	0.0

a_1	f_1 [kHz]	Φ_1 [rad]	a_2	f_2 [kHz]	Φ_2 [rad]
2.0	1.2	0.0	1.0	1.3	0.0

N	Fs [kHz]
	10

Questions:

2.4.1	Determine the record length N so that both frequency components f_1 and f_2 are located separately in the spectrum.
2.4.2	Print and discuss the result.
2.4.3	Which is the spacing Δf of the spectral lines?
2.4.4	Which frequencies are represented by the both highest spectral lines located closely? Mark the corresponding frequencies f_1 and f_2 on the frequency axis.

Exercise 2.5

Use the following parameters:

a_0	a_n
-1.0	0.0

a_1	f_1 [kHz]	Φ_1 [rad]	a_2	f_2 [kHz]	Φ_2 [rad]
2.0	1.2	0.0	1.0	1.3	$\pi/3$

N	Fs [kHz]
	10

Questions:

2.5.1	Determine the record length N so that both frequency components f_1 and f_2 are located separately in the spectrum.
2.5.2	Print and discuss the result.
2.5.3	Which is the spacing Δf of the spectral lines?
2.5.4	How can the spacing Δf between the spectral lines decreased ensuring a reliable resolution of two different signals with the frequencies f_1 and f_2.

Exercise 2.6

Use the following parameters:

a_0	a_n
-1.0	0.0

a_1	f_1 [kHz]	Φ_1 [rad]	a_2	f_2 [kHz]	Φ_2 [rad]
3.0	0.78125	0.0	0.3	1.055	$\pi/2$

N	Fs [kHz]
128	10

Questions:

2.6.1	Print and discuss the result.
2.6.2	Which is the spacing Δf of the spectral lines?
2.6.3	Mark the corresponding frequencies f_1 and f_2 on the frequency axis.
2.6.4	Determine how many periods of each component are collected?

Exercise 2.7

Use the following parameters:

a_0	a_n
-1.0	0.0

a_1	f_1 [kHz]	Φ_1 [rad]	a_2	f_2 [kHz]	Φ_2 [rad]
3.0	0.7422	0.0	0.3	1.055	$\pi/2$

N	Fs [kHz]
128	10

Questions:

2.7.1	Print and discuss the result.
2.7.2	Which is the spacing Δf of the spectral lines?
2.7.3	Mark the corresponding frequencies f_1 and f_2 on the frequency axis.
2.7.4	Determine how many periods of each component are collected?

Exercise 2.8

Use the following parameters:

a_0	a_n
-1.0	0.0

a_1	f_1 [kHz]	Φ_1 [rad]	a_2	f_2 [kHz]	Φ_2 [rad]
3.0	0.7422	0.0	0.3	1.055	$\pi/2$

N	Fs [kHz]
128	10

Questions:

2.8.1	Scale the N samples of the record with a so-called Hanning window function $w[n]=0.5\cdot\left(1-\cos\frac{2\pi n}{N}\right)$
2.8.2	Make a draft of the window function. How is the window looking we used until now?
2.8.3	Print and discuss the result. Which is the spacing Δf of the spectral lines?
2.8.4	Mark the corresponding frequencies f_1 and f_2 on the frequency axis.
2.8.5	Compare the computed spectrum with that from exercise 2.7.

Exercise 2.9

Use the following parameters:

a_0	a_n
0.0	8.0

a_1	f_1 [kHz]	Φ_1 [rad]	a_2	f_2 [kHz]	Φ_2 [rad]
0.0	0.0	0.0	0.0	0.0	0.0

N	Fs [kHz]
128	10

Questions:

2.9.1	Print and discuss the result.
2.9.2	Characterize the spectrum of noise.

Exercise 2.10

Use the following parameters:

a_0	a_n
-1.0	0.5

a_1	f_1 [kHz]	Φ_1 [rad]	a_2	f_2 [kHz]	Φ_2 [rad]
3.0	0.7422	0.0	0.3	1.055	$\pi/2$

N	Fs [kHz]
128	10

Questions:

2.10.1	Print and discuss the result.
2.10.2	Mark the corresponding frequencies f_1 and f_2 on the frequency axis.

Exercise 2.11

Use the following parameters:

a_0	a_n
-1.0	0.5

a_1	f_1 [kHz]	Φ_1 [rad]	a_2	f_2 [kHz]	Φ_2 [rad]
3.0	0.7422	0.0	0.3	1.055	$\pi/2$

N	Fs [kHz]
128	10

Questions:

2.11.1	Multiply the record set with the Hanning window function $w[n] = 0.5 \cdot \left(1 - \cos\frac{2\pi n}{N}\right)$
2.11.2	Print and discuss the result.
2.11.3	Compare the computed spectrum with that from exercise 2.10.

Exercise 2.12

In the last part the spectrum of a given recorded signal will be observed using different types of windows.

2.12.1	Firstly clear your workspace (Matlab command: clear all) and load the given Matlab data file *"Test_Signal.mat"* into your workspace (command *load*). **Note**: The signal consists of a DC-component, two real sinusoidal components and noise (normally distributed).
2.12.2	Create a new Matlab script file and calculate the spectrum of the signal by using the FFT-Algorithm (commands *fft* and *fftshift*). Choose a rectangular window and set N_{zp} = 8192 for the *fft*-command. Plot only the magnitude of the spectrum! Try to identify the amplitude of the DC-component and the (normalized) DT-frequency of the two sinusoidal components, if possible.
2.12.3	Repeat the previous exercise (2.12.2) for the following windows: • Bartlet Window • Hanning Window • Hamming Window • Blackman Window • Kaiser Window Compare the magnitude of all spectra and observe the effect of the different windows! **Notes**: Apply only the Kaiser Window if remaining time is not sufficient. Choose $\alpha = 10$ for the Kaiser Window and use the Matlab command *besseli (0,* α *)* to determine the modified Bessel function of order zero.